On an intrinsic approach of the guiding-center anholonomy and gyro-gauge-arbitrariness

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The guiding-center anholonomy and gyro-gauge arbitrariness are considered in the light of a recently proposed gauge-independent gyro-angle. The difficulties are shown to completely disappear in this coordinate system, directly given by the physical state, but to have regular counterparts, related to intrinsic properties of the system. The basic differential operators do not behave just as partial derivatives. Covariant gradients with non-zero commutators show up, as well as a connection arbitrariness, because the coordinate system is constrained. Within the traditional coordinate system, the difficulties are found to come from the requirement for the coordinates to fit with the basic derivative operators, which also explains the non-global existence of the standard gyro-angle coordinate.

To make covariant gradients commute, a connection is introduced for the pitch-angle as well, induced by its physical definition, but non-zero commutators are not avoided in phase-space. This provides an interesting framework to identify existence conditions for a global splitting of the momentum into scalar coordinates for both the pitch-angle and the gyro-angle, with conclusions extending previous results on the existence of a gauge-dependent gyro-angle.

Last, a coordinate system both gauge-independent and unconstrained is obtained by generalizing the method of intrinsic coordinates non-adapted to the basic derivative operators of the theory. The need for covariant derivatives is then removed and the scheme is simplified by eliminating both the gauge- or connection-arbitrariness and the non-zero commutators or anholonomy.

Key Words: Guiding-center, anholonomy, gyro-gauge, gauge-independent coordinates, constrained coordinates, connection on the circle, covariant derivative, principal circle bundle, commutator of vector fields.

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I. INTRODUCTION

The dynamics of a charged particle in a strong magnetic field shows up a separation of time scales which can be used to isolate completely the slow dynamics from the fast coordinate, the gyro-angle, which measures the Larmor gyration of the particle around the magnetic-field lines, and to build a conjugated constant of motion, the magnetic moment. This is achieved by the guiding-center reduction, which performs a perturbative change of coordinates in an expansion in the Larmor radius, to get a slow reduced motion and thus reduce by two the dimension of the effective dynamics[1–7]. It has wide applications in physics since most physical phenomena involve time scales and spatial scales much larger than the Larmor scales[8,9]. Especially, it is the starting point for the gyrokinetic model of plasma dynamics, which is a key model in the study of plasma micro-turbulence[10].

Among the guiding-center coordinates, the gyro-angle plays a leading role, but its definition raises several difficulties both from a mathematical and from a physical point of view[11–15], such as non-global existence, gyro-gauge dependence and anholonomy.

In recent works[16,17], we showed that both aspects of guiding-center theory (averaging reduction, and presence of the magnetic moment) can be addressed while using a physical coordinate for the gyro-angle, which is the unit vector of the component of the momentum perpendicular to the magnetic field. No gauge fixing was needed. It was shown in[18] that this intrinsic coordinate can be used also in the standard procedure for the guiding-center reduction, which provides a Hamiltonian structure for the guiding-center dynamics and a maximal reduction for the guiding-center Lagrangian. All the results of the literature can thus be obtained using this physical but constrained coordinate.

In the light of this intrinsic approach, we now come back to the difficulties of the usual coordinate. This can clarify whether they are artefacts of those coordinates, or related to intrinsic properties of the physics and mathematics of the system. In the latter case, it is interesting to study how these properties can be observed in the gauge-independent formulation, i.e. what the counterparts of the traditional difficulties are. Indeed, the guiding-center derivations in[16–18] showed that in the intrinsic approach, all the features of the traditional approach seemed to be present, including a kind of generalized gauge vector. This makes it necessary to clarify whether the gauge-independent coordinate actually resolves the issues or only transfers them into other issues.

An additional advantage of such a study is to make clearer the essential differences between the two coordinate systems. This will suggest generalizations of the intrinsic approach, either to provide it with a scalar gyro-angle coordinate, or to release it from the presence of constrained coordinates.

The paper is organized as follows. In sect. 2, the standard and the gauge-independent gyro-angles are reminded, and it is emphasized that when using this last variable, the traditional difficulties associated with the gyro-angle are absent from the coordinate system. In sect. 3, the counterpart of the gauge arbitrariness in the intrinsic approach is studied; it will be related to a connection arbitrariness for covariant gradients. In sect. 4, we turn to the anholonomy question, which will be related to non-zero commutators between covariant gradients. In sect. 5, connections removing non-zero commutators of gradients or more generally of the basic derivative operators are considered, which will provide an interesting approach of the existence condition for a scalar gyro-angle coordinate. In sect. 6, the flaws raised by the presence of a constrained coordinate are addressed, and it is shown that they can be eliminated by avoiding the splitting between the gyro-angle and the pitch-angle in the coordinate system.

II. LOCAL AND GLOBAL COORDINATES FOR THE GYRO-ANGLE

The physical system under consideration is a charged particle with position $\mathbf{q}$, momentum $\mathbf{p}$, mass m and charge e, under the influence of an electromagnetic magnetic field $(\mathbf{E}, \mathbf{B})$. The motion is given by the Lorentz force

$$\dot{\mathbf{q}} = \tfrac{\mathbf{p}}{m}\,,$$
$$\dot{\mathbf{p}} = \tfrac{\mathbf{p}}{m} \times e\mathbf{B} + e\mathbf{E}\,.$$

When the magnetic field is strong, the motion implies a separation of time scales. This is best seen by choosing convenient coordinates for the momentum space, for instance[17]

$$\begin{aligned} p &:= \|\mathbf{p}\|\,, \\ \varphi &:= \arccos\left(\tfrac{\mathbf{p}\cdot\mathsf{b}}{\|\mathbf{p}\|}\right)\,, \\ \mathsf{c} &:= \tfrac{\mathbf{p}_\perp}{\|\mathbf{p}_\perp\|}\,, \end{aligned} \tag{1}$$

where $\mathsf{b} := \frac{\mathbf{B}}{\|\mathbf{B}\|}$ is the unit vector of the magnetic field, and $\mathbf{p}_\perp := \mathbf{p} - (\mathbf{p}\cdot\mathsf{b})\mathsf{b}$ is the so-called *perpendicular momentum*, i.e. the orthogonal projection of the momentum onto the plane perpendicular to the magnetic field. The coordinate p is the norm of the momentum; the coordinate φ is the so-called *pitch-angle*, i.e. the angle between the velocity and the magnetic field. The last variable c is the unit vector of the perpendicular momentum.

Then, the equations of motion write

$$\begin{aligned}
\dot{\mathbf{q}} &= \tfrac{\mathbf{p}}{m}\,, \\
\dot{p} &= \tfrac{e\mathbf{E}\cdot\mathbf{p}}{p}\,, \\
\dot{\varphi} &= -\tfrac{\mathbf{p}}{m}\cdot\nabla\mathsf{b}\cdot\mathsf{c} + \tfrac{e\mathbf{E}}{p\sin\varphi}\cdot\left(\cos\varphi\,\tfrac{\mathbf{p}}{p} - \mathsf{b}\right)\,, \\
\dot{\mathsf{c}} &= -\tfrac{eB}{m}\mathsf{a} - \tfrac{\mathbf{p}}{m}\cdot\nabla\mathsf{b}\cdot(\mathsf{cb} + \mathsf{aa}\cot\varphi) + \tfrac{e\mathbf{E}\cdot\mathsf{a}}{p\,\sin\varphi}\mathsf{a}\,,
\end{aligned} \tag{2}$$

where $\mathbf{p}$ is now a shorthand for $p(\mathsf{b}\cos\varphi + \mathsf{c}\sin\varphi)$, B is the norm of the magnetic field and, following Littlejohn's notations[5,6], the vector $\mathsf{a} := \mathsf{b}\times\mathsf{c}$ is the unit vector of the Larmor radius, so that $(\mathsf{a},\mathsf{b},\mathsf{c})$ is a right-handed orthonormal frame (rotating with the momentum).

In the case of a strong magnetic field, the only fast term, the Larmor frequency $\omega_L := \frac{eB}{m}$, corresponds to the gyration of the particle momentum around the magnetic field. It concerns only one coordinate, c, the direction of the perpendicular momentum $\mathbf{p}_\perp$ in the 2-dimensional plane perpendicular to the magnetic field. It corresponds to the gyro-angle.

To get a scalar angle instead of the vector c, one chooses at each point $\mathbf{q}$ in space a direction $\mathsf{e}_1(\mathbf{q}) \in \mathbf{B}^\perp(\mathbf{q})$ in the plane perpendicular to the magnetic field, which will be considered as the reference axis. Then, the angle θ is defined as the oriented angle between the chosen reference axis $\mathsf{e}_1(\mathbf{q})$ and the vector c through the following relation :

$$\mathsf{c} = -\mathsf{e}_1\sin\theta - \mathsf{e}_2\cos\theta\,, \tag{3}$$

with $\mathsf{e}_2 := \mathsf{b}\times\mathsf{e}_1$ the unit vector such that $(\mathsf{b},\mathsf{e}_1,\mathsf{e}_2)$ is a (fixed) right-handed orthonormal frame[5]. The angle θ is the usual coordinate for the gyro-angle[1,3,5,7]. Its equation of motion is

$$\dot{\theta} = \tfrac{eB}{m} + \cot\varphi\tfrac{\mathbf{p}}{m}\cdot\nabla\mathsf{b}\cdot\mathsf{a} + \tfrac{\mathbf{p}}{m}\cdot\nabla\mathsf{e}_1\cdot\mathsf{e}_2 - \tfrac{e\mathbf{E}\cdot\mathsf{a}}{p\,\sin\varphi}\,. \tag{4}$$

From the initial dynamics $(\dot{\mathbf{q}},\dot{p},\dot{\varphi},\dot{\theta})$, guiding-center reductions perform a change of coordinates $(\mathbf{q},p,\varphi,\theta) \longrightarrow (\bar{\mathbf{q}},\bar{p},\bar{\varphi},\bar{\theta})$, to obtain a reduced dynamics with suitable properties, mainly a constant of motion $\dot{\bar{p}} = 0$, and a slow reduced motion $(\dot{\bar{\mathbf{q}}},\dot{\bar{\varphi}})$ that is both

independent of the fast coordinate $\bar{\theta}$ and Hamiltonian[7,18]. The reduced position $\bar{\mathbf{q}}$ is the *guiding-center*, and the constant of motion $\bar{p}$ is the *magnetic moment*; it is close to the well-known adiabatic invariant $\mu := \frac{\mathbf{p}_\perp^2}{2mB}$, and is usually written $\bar{\mu}$ instead of $\bar{p}$.

In the definition of θ, the necessary introduction of e_1 implies important and awkward features in the theory.

First, the choice of e_1 is arbitrary, which induces a local gauge in the theory. The coordinate system is gauge dependent, since the value of θ depends of the chosen $\mathsf{e}_1(\mathbf{q})$. For a general reduction procedure, the guiding-center dynamics can end up gauge-dependent. For instance, the maximal reduction by Lie-transforming the phase-space Lagrangian is gauge dependent[18].

Guiding-center reductions have to use prescriptions in order to avoid such unphysical results. For instance, in the reduced Lagrangian, the 1-form $\mathrm{d}\theta$ must appear only through the quantity $d\theta - (d\mathbf{q}{\cdot}\nabla\mathsf{e}_1){\cdot}\mathsf{e}_2$[6,7]. These gauge-dependence issues emphasize that the gyro-angle is artificial, it is not given by the physics and its meaning is restricted.

Second and more substantial, a continuous choice of e_1 does not exist globally in a general magnetic geometry[14,15]. This is easily explained because the possible values for e_1 define a circle bundle over the configuration space[2,15], and a specific choice e_1 is a global section of the bundle, which corresponds a trivialization of the bundle, whose coordinate system is precisely $(\mathbf{q}, \theta)$. But such a trivialization does not exist globally for a general circle bundle. In the case of the guiding-center, this global non-existence can be proven by using the theory of principal bundles and characteristic classes[15]. Thus, the gyro-angle does not exist in the whole physical system in general, which means that it does not capture the mathematical description of the system, but only a strongly simplified description, valid only in the trivial case.

It was mentioned in[15] that the local descriptions agree with a global description provided the change of local descriptions satisfy some relations; for instance, it explains why the 1-form $d\theta$ must appear only through the combination $d\theta - (d\mathbf{q}{\cdot}\nabla\mathsf{e}_1){\cdot}\mathsf{e}_2$. This means that working with θ is not meaningless; for instance, it indeed gives a slow guiding-center dynamics $(\dot{\bar{\mathbf{q}}}, \dot{\bar{\varphi}})$ that is globally defined, but it does not imply that the local coordinate θ has a global meaning, neither does it explain what the global description of the gyro-angle is.

Last, even the local description is not completely regular, because it involves a non-

holonomic phase in the gyro-angle. When a loop γ is performed in position space, then at the end of the process, all the physical quantities have recovered their value, but the variation of the gyro-angle involves a contribution (which is related to the third term in the right-hand side of Eq. (4))

$$\Delta\theta_g := \oint_\gamma (d\mathbf{q}\cdot\nabla\mathsf{e}_1)\cdot\mathsf{e}_2\,,$$

called the geometric phase, which is not zero[5,11,12]. This is the well-known anholonomy issue of the gyro-angle coordinate.

A similarity with Berry's phase and more generally with Hannay's phase was often pointed out[5,11,19], but there are significant differences, as mentioned in[11]. Especially, these phases are related to adiabaticity, with a single path in parameter space followed by the system, which makes them physically determinable. On the contrary, guiding-center anholonomy is related to path dependence in configuration space, with all paths coexisting simultaneously. This precludes any definite value for this phase at any point in the system, which raises questions whether this phase is physically meaningful or if it affects only non-physical quantities concerning the extrinsic coordinate system.

All these issues come because θ is only an artificial quantity. They motivated to keep the primitive gauge-independent coordinate c instead of introducing θ. Even if this quantity does not have scalar values, it embodies all the same an angle, since it is a unit vector in a plane and hence belongs to a circle $\mathbb{S}^1$.

It does represent the physical quantity corresponding to the gyro-angle: θ never appears by itself in the theory (e.g. in guiding-center transformations), except in its own definition and subsequent relations; what appears everywhere is the vector c[3,5,7]; even the correction to the gyro-angle $\bar{\theta} - \theta$ in guiding-center reductions does not involve θ by itself, but only through c. A simple illustration of this argument, as well as of the gauge-dependence or not, can be found in the equations of motion (2) and (4).

The coordinate c is globally defined, since the perpendicular momentum is well defined everywhere. It is useful to remind that the points where the momentum is parallel to the magnetic field are always implicitly excluded from guiding-center theory, even in the local description using the scalar coordinate θ. Indeed, at those points, the angle θ can not be defined. In addition, guiding-center expansions involves many $\sin\varphi$ in denominators, which

implies to exclude the mentioned points.

The variations of the variable c do not include the gauge contribution $(d\mathbf{q}\cdot\nabla\mathsf{e}_1)\cdot\mathsf{e}_2$, with its anholonomic geometric phase. After a loop in configuration or phase space, when all physical quantities come back to their initial value, so will the contributions to the quantity c, since it is directly related to the particle state.

Last, the coordinate c agrees with the mathematical description of the system. For any magnetic geometry, it induces a circle bundle: the circle for c is position-dependent since c is perpendicular to the magnetic field, which means that c is not just in $\mathbb{S}^1$, but in $\mathbb{S}^1(\mathbf{q})$. A few consequences will be studied in the next sections. In the traditional coordinate system, this picture is absent, since θ is independent of the position; the circle bundle rather concerns the vector e_1, but the corresponding bundle is different from the intrinsic bundle for the gyro-angle, since it is not defined by the whole phase space, but confined to the four-dimensional space $(\mathbf{q}, \mathsf{e}_1)$. In addition, the global section e_1 assumes the topology of the bundle trivial.

Accordingly, the use of the gyro-angle c removes from the coordinate system all of the issues involved in the traditional gyro-angle θ. It indeed provides the intrinsic description of the physics and mathematics of the system.

III. GAUGE ARBITRARINESS AND CONNECTION FREEDOM

When using the coordinate c, the spatial dependence of $\mathbb{S}^1$ implies that the coordinate space is constrained, i.e. the coordinates are not independent of each other. When the position $\mathbf{q}$ is changed, the coordinate c cannot be kept unchanged, otherwise it may get out of $\mathsf{b}^\perp$:

$$\nabla\mathsf{c} \neq 0\,.$$

Differentiating Eq. (1) with respect to $\mathbf{q}$, we find[17]

$$\nabla\mathsf{c} = -\nabla\mathsf{b}\cdot(\mathsf{c}\mathsf{b} + \mathsf{a}\mathsf{a}\cot\varphi)\,, \tag{5}$$

which is well defined everywhere, since the points where $\cot\varphi = \pm\infty$, i.e. where $\mathbf{p}$ is parallel to the magnetic field $\mathbf{B}$, are excluded from the theory.

Eq. (5) must not be given a completely intrinsic meaning, because the two terms in its right-hand side play a very different role through coordinate change: the first term is always

unchanged, whereas the second one is generally changed. For instance, if one uses the scalar angle θ as a local coordinate for c, then Eq. (5) becomes

$$\nabla\mathsf{c} = -\nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b} + \mathbf{R}\ \mathsf{a}\,, \tag{6}$$

where

$$\mathbf{R} := \nabla\mathsf{e}_1\cdot\mathsf{e}_2$$

is the so-called *gauge-vector*. It is a function of the position $\mathbf{q}$, and is not unique: it depends of the choice of gauge $\mathsf{e}_1(\mathbf{q})$.

The reason for this difference of role is that the definition space $\mathbb{S}^1(\mathbf{q})$ for the coordinate c imposes exactly the first term in Eq. (5), but gives no constraints on the second term. Indeed, the gyro-angle c is a free 1-dimensional coordinate, but it is at the same time a vector, immersed in $\mathbb{R}^3$, and the two remaining dimensions are fixed by the condition for c to have unit norm $\mathsf{c}\cdot\mathsf{c} = 1$ and to be transverse to the magnetic field $\mathsf{c}\cdot\mathsf{b} = 0$. This implies

$$\nabla\mathsf{c}\cdot\mathsf{c} = 0 \quad \text{and} \quad \nabla\mathsf{c}\cdot\mathsf{b} = -\nabla\mathsf{b}\cdot\mathsf{c}\,.$$

Thus, in $\nabla\mathsf{c}$, only the component parallel to a is not imposed by intrinsic properties linked with $\mathbb{S}^1(\mathbf{q})$. It is induced by the specific definition chosen for the gyro-angle coordinate, but from an intrinsic point of view, it is completely free:

$$\nabla\mathsf{c} = -\nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b} + \mathbf{R}_g\ \mathsf{a}\,, \tag{7}$$

where

$$\mathbf{R}_g := \nabla\mathsf{c}\cdot\mathsf{a}$$

is a free function of the phase-space.

The geometric picture of this freedom is the following. In the gradient $\nabla\mathsf{c}$, i.e. in the effects of an infinitesimal spatial displacement, one of the terms, $-\nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b}$, is mandatory, since it is necessary and sufficient for c to remain perpendicular to b in the process of spatial transportation; this is easily seen on a diagram. The other term is only optional. It corresponds to a rotation of c around b (hence a gyration around the circle) accompanying the spatial displacement, but it could be removed, or given a different value. It is not imposed by intrinsic properties and has to be arbitrarily chosen, in a similar way as when the target set of a projection is determined, but not the kernel. It corresponds to the way points in a

circle $\mathbb{S}^1(\mathbf{q}_1)$ at the position $\mathbf{q}_1$ are "projected" (more precisely connected) to points in the circle $\mathbb{S}^1(\mathbf{q}_1+\delta\mathbf{q}_1)$ at a neighbouring position. This exactly corresponds to the connection in the circle bundle. As a result, although the gyro-gauge with its arbitrariness is absent from the intrinsic approach, some arbitrariness is present, not in the coordinate system itself, but in the choice of connexion for the covariant derivative.

The comparison with the gauge-dependent approach emphasizes the relationship between the two arbitrarinesses. Eq. (6) shows that the connection is then embodied in the gauge vector $\mathbf{R}$. This quantity is called gauge vector because it exactly embodies the information on the local gauge. Changing the gauge $\mathsf{e}_1(\mathbf{q}) \longrightarrow \mathsf{e}'_1(\mathbf{q})$ just means adding to the value of θ a scalar function $\psi(\mathbf{q})$, whose value is the angle between e_1 and e'_1 at $\mathbf{q}$[5,11]. Under this change, the gauge vector is changed by

$$\mathbf{R}' = \mathbf{R} + \nabla\psi \,. \tag{8}$$

The term $\nabla\psi$ exactly embodies the purely local gauge. The remaining part of the gauge corresponds to a global gauge, i.e. the value of ψ at a reference point $\mathbf{q}_1$; changing it corresponds to a global rotation by the same angle $\psi(\mathbf{q}_1)$ for all points $\mathbf{q}$; it is not specifically a (local) gauge, but rather a common change of angular coordinate.

As a side comment, which will be useful for the following, Eq. (8) shows that even if the gauge vector is gauge dependent, its curl is not, it is related to intrinsic properties of the system[5]

$$\nabla\times\mathbf{R} = \mathbf{N} \,, \tag{9}$$

$$\text{with}\quad \mathbf{N} := \tfrac{\mathsf{b}}{2}\big(\mathrm{Tr}(\nabla\mathsf{b}\cdot\nabla\mathsf{b}) - (\nabla\cdot\mathsf{b})^2\big) + (\nabla\cdot\mathsf{b})\mathsf{b}\cdot\nabla\mathsf{b} - \mathsf{b}\cdot\nabla\mathsf{b}\cdot\nabla\mathsf{b} \,. \tag{10}$$

Accordingly, the gauge vector can not be a free function of the position space, but it must satisfy the integrability condition (9). Especially, the choice $\mathbf{R} = 0$ is not available in general, even locally.

With the interpretation above for $\mathbf{R}$, Eq. (6) means that the local gauge freedom is exactly the counterpart of the connection freedom $\mathbf{R}_g$. The geometric reason is that θ is the angle between e_1 and c; through spatial transportation $\delta\mathbf{q}$ with θ unchanged, the gauge vector $\mathbf{R}$

means that e_1 is rotated around b by an angle $\delta\mathbf{q}\cdot\mathbf{R}$, which implies that c undergoes the same rotation. Thus, the connection for e_1 is at the same time a connection for c.

It explains why the gauge arbitrariness originated from an inherent property of the system, but it makes more precise the meaning and the content of the intrinsic arbitrariness: first, it does not represent an arbitrariness in the coordinate system, but a choice in the covariant gradient; second, it implies not only a restricted class of functions of the configuration space but a free function of the phase space.

The first point clarifies how in the intrinsic formulation, the arbitrariness is present, but it does not affect the coordinates at all, whereas in the standard formulation, it affects also the coordinate θ. This is illustrated by the guiding-center transformation: for the gyro-angle c, the transformation is connection-independent[17,18]; for the coordinate θ, the transformation $\bar{\theta} = \cdot\cdot e^{\mathbf{G}_2} e^{\mathbf{G}_1}\theta$ is gauge-dependent, but in such a way as to make the induced transformation $\bar{\mathsf{c}} = \cdot\cdot e^{\mathbf{G}_2} e^{\mathbf{G}_1}\mathsf{c}$ for c gauge independent, with $\mathbf{G}_n$ the vector field generating the n-th order transformation; for instance, the first-order reduced gyro-angle $\bar{\theta}$ is given by $\mathbf{G}_1^\theta$, which is gauge-dependent, but the first-order $\bar{\mathsf{c}}$ is given by $(\mathbf{G}_1^{\mathbf{q}}\cdot\nabla + \mathbf{G}_1^\theta\ \partial_\theta)\mathsf{c}$, which is gauge-independent[5–7]. Another example is given by the guiding-center Poisson bracket; when using the coordinate θ, its expression is gauge-dependent because of the presence of the gauge-vector $\mathbf{R}$[5,6]; it was noticed that this presence can be combined with the gradient through the combination $\nabla + \mathbf{R}\ \partial_\theta$[10]; actually, this is exactly the connection-independent gradient for the coordinate c; indeed, formally the covariant gradient can be written $\nabla = \partial_{\mathbf{q}|\mathsf{c}} + \nabla\mathsf{c}\cdot\partial_{\mathsf{c}|\mathbf{q}}$, which is connection-dependent, but in such a way as to make the quantity $\nabla - \nabla\mathsf{c}\cdot\partial_{\mathsf{c}|\mathbf{q}} = \nabla + \nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b}\cdot\partial_\mathsf{c} - \mathbf{R}_g\mathsf{a}\cdot\partial_\mathsf{c}$ connection-independent; this last quantity is not a gradient because the second term in the right-hand side brings c out of its definition space; thus, this term has to be removed to obtain the connection-independent gradient $\nabla_\circ := \nabla + \mathbf{R}_g\ \partial_\theta$; it is the minimal connection, since it formally writes $\nabla_\circ = \partial_{\mathbf{q}|\mathsf{c}} - \nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b}\cdot\partial_{\mathsf{c}|\mathbf{q}}$, which corresponds to $\nabla_\circ\mathsf{c} = -\nabla\mathsf{b}\cdot\mathsf{c}\ \mathsf{b}$, or equivalently $(\mathbf{R}_g)_\circ = 0$. This minimal connection corresponds to the orthogonal projection, in the picture mentioned above where the connection is viewed as a projection from the circle $\mathbb{S}^1(\mathbf{q})$ to the circle $\mathbb{S}^1(\mathbf{q}+\delta\mathbf{q})$.

The second point can have practical consequences. For instance, the physical connection (5) is not available when using the traditional coordinate θ, because it does not correspond to a gauge fixing (since it depends on the momentum variables), neither does the connection $\mathbf{R} = 0$, which would simplify computations and remove the arbitrary terms from the

theory. As a result, there is no natural gauge fixing in this case, and previous works tried to identify a natural fixing based on the magnetic geometry[11,12]. On the contrary, with the gauge-independent coordinate c, the physical choice $\mathbf{R}_g = -\cot\varphi \nabla\mathsf{b}\cdot\mathsf{a}$ does correspond to a connection fixing, and the simplifying choice $\mathbf{R}_g = 0$ is just the minimal connexion, as well as the natural geometrical connexion, i.e. the one which is induced by the definition of $\mathbb{S}^1(\mathbf{q})$ and the one that corresponds to the connection-independent gradient, as shown in the previous paragraph.

Thus, the intrinsic approach is not just an optional reformulation of the theory. It emphasizes the intrinsic properties underlying the gauge-arbitrariness, which rather concern the covariant gradients than the coordinate system, and it makes available some relevant gradients which were inaccessible in the gauge-dependent formulation.

IV. ANHOLONOMY AND NON-ZERO COMMUTATORS

Also the anholonomy of the gauge-dependent approach has a counterpart in the intrinsic approach. The initial phenomenon is that the variations of θ do not depend only on the state of the particle, but also of the gauge-fixing e_1. This last contribution is the geometric phase, and it is not zero after a closed loop γ in configuration space:

$$\Delta\theta_g = \oint_\gamma (d\mathbf{q}\cdot\nabla\mathsf{e}_1)\cdot\mathsf{e}_2 = \int_S \nabla\times\mathbf{R}\cdot d\mathbf{S} \neq 0\,, \tag{11}$$

where S is a surface with boundary $\partial S = \gamma$. As the integrand is given by Eq. (10), it is gauge-independent and the anholonomy term (11) can not be made zero by a choice of gauge. It suggests that it corresponds to an intrinsic property of the system; it is why this issue was considered as unavoidable in guiding-center coordinates[5,11].

With the gyro-angle c, the anholonomy is absent from the coordinate system, since the coordinates are defined directly from the physical state. There is no extrinsic quantity (such as e_1) implied in the definition of this gyro-angle to generate anholonomy.

However, c is not a scalar angle, it is a vector, and its infinitesimal variations are two-dimensional, as shown in Eq. (7). For an intrinsic description of the phenomenon at work in Eq. (11), it is convenient to identify a scalar quantity for the variation of the gyro-angle c.

The change of c in the direction b is not relevant, since it just corresponds to maintaining

c in its definition space through spatial displacement. Thus, the effective variation of the gyro-angle is only in the a direction; more precisely, it corresponds to the change of c after removing the contribution coming from the spatial displacement; from this point of view,

$$\delta\Theta := -\mathsf{a}\cdot(d\mathsf{c} - d\mathbf{q}\cdot\nabla\mathsf{c}) = -\mathsf{a}\cdot d\mathsf{c} + d\mathbf{q}\cdot\mathbf{R}_g$$

is the quantity measuring the Larmor gyration; the minus sign in the prefactor is a convention in order to agree with the usual orientation for the gyro-angle θ.

The relevance of the scalar variation $\delta\Theta$ is emphasized by the set of 1-forms

$$\Big(d\mathbf{q}, dp, d\varphi, \delta\Theta\Big) \tag{12}$$

being the dual basis to the set of vector fields

$$\Big(\nabla, \partial_p, \partial_\varphi, \partial_\theta\Big)\,,$$

which are the natural derivative operators of the theory. Here the generator of Larmor gyration is written ∂_θ, but its definition does not depend on the gauge:

$$\partial_\theta := -\mathsf{a}\cdot\partial_{\mathsf{c}}\,.$$

In addition, when the local gauge-dependent description for the gyro-angle is used, it is readily checked that $\delta\Theta = d\theta$, which confirms that $\delta\Theta$ is the intrinsic (global) quantity corresponding to $d\theta$. This explains why the 1-form $d\theta$ is gauge-dependent, and the associated gauge-independent 1-form is $d\theta - d\mathbf{q}\cdot\mathbf{R}$, since $\delta\Theta$ depends on $\mathbf{R}_g$, and the associated connexion-independent quantity is $-\mathsf{a}\cdot d\mathsf{c} = \delta\Theta - d\mathbf{q}\cdot\mathbf{R}_g$. An essential difference compared to $d\theta$ is that $\delta\Theta$ is not closed:

$$d(\delta\Theta) = -(d\mathbf{q}\cdot\nabla\mathsf{b})\cdot(\mathsf{b}\times\mathsf{b}'d\mathbf{q}) + d\mathbf{R}_g\cdot\wedge\ d\mathbf{q}\,, \tag{13}$$

where for notational convenience, the primed notation was used for gradients acting on their left: $\mathsf{b}'d\mathbf{q} = d\mathbf{q}\cdot\nabla\mathsf{b}$. The wedge symbol $\wedge$ indicates antisymmetry: $a.\wedge\ b = a.b - b.a$.

The variation of Θ is defined along a path γ by $\Delta\Theta = \int_\gamma \delta\Theta$. After performing one closed path, it is given by:

$$\Delta\Theta = \oint_\gamma \delta\Theta = \int_S d(\delta\Theta) \neq 0\,, \tag{14}$$

which is non-zero in general. Thus, Θ is not an holonomic quantity.

More precisely, using Eq. (13), the anholonomy (14) of the scalar angle can be written

$$\Delta\Theta = \Delta\Theta_B + \Delta\Theta_c\,,$$

where the first contribution comes from the magnetic geometry

$$\Delta\Theta_B = -\int_S (d\mathbf{q}\cdot\nabla\mathsf{b})\cdot(\mathsf{b}\times\mathsf{b}'d\mathbf{q})\,, \tag{15}$$

and the second contribution comes from the choice of connection

$$\Delta\Theta_c = \int_S d\mathbf{R}_g\cdot\wedge\ d\mathbf{q} \tag{16}$$

and is expected to be the intrinsic counterpart of the geometric phase.

When the connection is given its natural geometrical definition $\mathbf{R} - g = 0$, then its contribution given by Eq. (16) is zero, and the gyro-angle anholonomy is exactly given by the anholonomic term (15) due to the magnetic geometry.

On the other hand, when the local gauge-dependent coordinate θ is used for the gyro-angle, the connection vector $\mathbf{R}_g = \mathbf{R} = \nabla\mathsf{e}_1\cdot\mathsf{e}_2$ depends only on $\mathbf{q}$, and the integrand of its contribution (16) exactly compensates the magnetic contribution (15), since it writes

$$\begin{aligned} d\mathbf{R}\cdot\wedge\ d\mathbf{q} &= (d\mathbf{q}\cdot\nabla\mathsf{e}_2)\cdot\wedge\ (d\mathbf{q}\cdot\nabla\mathsf{e}_1) \\ &= (d\mathbf{q}\cdot\nabla\mathsf{b})\cdot(\mathsf{e}_2\mathsf{e}_1 - \mathsf{e}_1\mathsf{e}_2)\cdot(\mathsf{b}'d\mathbf{q}) \\ &= (d\mathbf{q}\cdot\nabla\mathsf{b})\cdot\mathsf{b}\times(\mathsf{b}'d\mathbf{q})\,, \end{aligned} \tag{17}$$

where the first equality comes from the antisymmetry, and the second comes by inserting the identity matrix $(\mathsf{bb} + \mathsf{e}_1\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_2)\cdot$ and by using that

$$\begin{aligned} (d\mathbf{q}\cdot\nabla\mathsf{e}_1)\cdot\mathsf{e}_1 &= 0\,, \\ (d\mathbf{q}\cdot\nabla\mathsf{e}_2)\cdot\mathsf{e}_2 &= 0\,, \\ (d\mathbf{q}\cdot\nabla\mathsf{e}_i)\cdot\mathsf{b} &= -(d\mathbf{q}\cdot\nabla\mathsf{b})\cdot\mathsf{e}_i\,, \end{aligned}$$

because $(\mathsf{b}, \mathsf{e}_1, \mathsf{e}_2)$ is an orthonormal basis.

For a comparison with the anholonomic phase (11), the integrand of the connection contribution (16), which is only a function of the position, can also be written

$$\begin{aligned} d\mathbf{R}\cdot \wedge\ d\mathbf{q} &= d\mathbf{q}\cdot(\nabla\mathbf{R} - \mathbf{R}')\ d\mathbf{q} \\ &= -d\mathbf{q}\cdot(\nabla\times\mathbf{R})\times d\mathbf{q}\,, \end{aligned}$$

which indeed agrees with (11).

These results can be shown to agree with Littlejohn's expression (10), using the antisymmetry of the matrix $\nabla\mathsf{b}\cdot\mathsf{b}\times\mathsf{b}'$ to write it as a cross product:

$$\begin{aligned} (d\mathbf{q}\cdot\nabla\mathsf{b})\cdot(\mathsf{b}\times\mathsf{b}'d\mathbf{q}) &= d\mathbf{q}^i\nabla^i\mathsf{b}\cdot\mathsf{b}\times\nabla^j\mathsf{b}\ d\mathbf{q}^j \\ &= d\mathbf{q}^i d\mathbf{q}^j\tfrac{1}{2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})\nabla_k\mathsf{b}\cdot\mathsf{b}\times\nabla_l\mathsf{b} \\ &= d\mathbf{q}^i d\mathbf{q}^j\tfrac{1}{2}\epsilon_{ijA}\epsilon_{Akl}\nabla_k\mathsf{b}\cdot\mathsf{b}\times\nabla_l\mathsf{b} \\ &= \epsilon_{iAj}d\mathbf{q}^i\left(-\tfrac{1}{2}\right)\epsilon_{Akl}\nabla_k\mathsf{b}\cdot\mathsf{b}\times\nabla_l\mathsf{b}\ d\mathbf{q}^j \\ &= d\mathbf{q}\cdot\mathbf{N}\times d\mathbf{q}\,, \end{aligned} \tag{18}$$

with

$$\begin{aligned} \mathbf{N}_A := &\left(-\tfrac{1}{2}\right)\epsilon_{Akl}\epsilon_{\alpha\beta\gamma}\nabla_k\mathsf{b}_\alpha\mathsf{b}_\beta\nabla_l\mathsf{b}_\gamma \\ = &\tfrac{\mathsf{b}_A}{2}\left(\mathrm{Tr}(\nabla\mathsf{b}\cdot\nabla\mathsf{b}) - (\nabla\cdot\mathsf{b})^2\right) \\ &+ (\nabla\cdot\mathsf{b})\ (\mathsf{b}\cdot\nabla\mathsf{b})_A - (\mathsf{b}\cdot\nabla\mathsf{b}\cdot\nabla\mathsf{b})_A\,, \end{aligned}$$

which is exactly Littlejohn's expression Eq. (10). In this computation, the first equality (18) comes from the antisymmetry of the matrix $\nabla\mathsf{b}\cdot\mathsf{b}\times\mathsf{b}'$, and all the other equalities are properties of the Levi-Civitá symbol. An alternative (more direct but heavier) way of proving the result is to insert the identity matrix $(\mathsf{bb}+\mathsf{e}_1\mathsf{e}_1+\mathsf{e}_2\mathsf{e}_2)\cdot$ everywhere in $(d\mathbf{q}\cdot\nabla\mathsf{b})\cdot(\mathsf{b}\times\mathsf{b}'d\mathbf{q})$; then expanding the formula and simplifying it gives the expected result.

So, in the local gauge-dependent case, the connection contribution is exactly the geometric phase

$$\Delta\Theta_c = \Delta\theta_g\,,$$

which confirms that $\Delta\Theta_c$ is the intrinsic counterpart of the geometric phase. In addition, it exactly compensates the integrand of the anholonomic magnetic contribution (15):

$$\Delta\Theta_c = -\Delta\Theta_B\,,$$

which explains both that the corresponding gyro-angle θ is holonomic $d(\delta\Theta) = d^2\theta = 0$, and that the associated geometric phase $\delta\theta_g$ is anholonomic.

The introduction of a scalar variation for c was used only to identify the correspondence between the local and global descriptions. From an intrinsic point of view, $\delta\Theta$ is not closed and hence Θ is not a proper coordinate, but it is not needed, since the gyro-angle coordinate is the vectorial quantity c. All the same, anholonomy effects can be viewed even in this intrinsic framework, either in the properties of the basic 1-forms (12) of the theory, as we saw previously, either more basically in the properties of the elementary vector fields of the theory. Indeed, the effects of an infinitesimal loop in configuration space are evaluated with the commutator of gradients[20]. Here, it is easily computed as

$$[\nabla_i, \nabla_j] = \Big(\nabla_i \mathsf{b} {\cdot} \mathsf{b} \times \nabla_j \mathsf{b} - \nabla_i (\mathbf{R}_g)_j + \nabla_j (\mathbf{R}_g)_i\Big)\, \partial_\theta \,, \tag{19}$$

which is a dual formula to Eq. (13).

The non-commutation of gradients in Eq. (19) is a consequence of the presence of a constrained coordinate system, with its associated non-zero connection, which means that the action of gradients does not fit directly with the coordinate system. After a closed path in the sense of the gradients, i.e. of the sum of infinitesimal variations, the coordinates do not recover their initial value. Conversely, after one loop in coordinate space, the coordinates recover their initial value, but the sum of infinitesimal changes is not zero. Thus, not only the anholonomy is inherent to the introduction of a scalar angle, which generalizes the conclusion of[5,11], but the anholonomy is an intrinsic feature of the space of particle states $(\mathbf{q}, p, \varphi, \mathsf{c})$. This feature, although absent from trivial coordinate systems, is not an issue. It is quite common in spaces with non-trivial geometry, e.g. in general relativity.

The issue comes with the gauge-dependent approach, when requiring a coordinate system that fits with the action of gradients, which is possible only when the geometry of the bundle trivial and which does not settle the anholonomy question. As a result, the scalar coordinate θ for the gyro-angle has holonomic (commuting) gradients but is local and involves anholonomic (unphysical) phases; on the contrary, the global gyro-angle c has anholonomic gradients rather than anholonomic phases.

This necessary alternative comes from Eq. (13) or (19), which shows that the anholonomy is unavoidable and intrinsically related to the magnetic geometry, through the anholonomic

term $\nabla\mathsf{b}\cdot\mathsf{b}\times\mathsf{b}'$, which has to be put either as a non-zero commutator of gradients, or as an anholonomic phase for the gyro-angle.

V. CONNECTIONS WITH NO NON-ZERO COMMUTATOR

One additional point is to be clarified: when the connection is given its physical expression (5), then the gyro-angle c is just the perpendicular velocity, which should be holonomic (i.e. it should not have non-zero commutators of gradients), since it is directly given by the physical momentum and the magnetic field, both of which are holonomic. The reason for this anholonomy is that the connexion was defined through the physical definition of c, but not the physical definition of the whole momentum. To do so, a more general connection should be used, which would concern also the pitch-angle φ.

The variable φ is not a constrained coordinate, since it is defined over an independent space $\mathbb{R}^1$. Unlike the variable c, it does not have to change value through spatial transportation, but it is allowed to. A flat (zero) connexion is possible but it is only the trivial choice, analogous to the choice $\mathbf{R}_g = 0$ for the coordinate c. In the same way as the free term $\mathbf{R}_g$, the coordinate φ can have an arbitrary connexion. Especially, its definition from the physical momentum through Eq. (1) induces a non-zero connexion

$$\nabla\varphi = -\nabla\mathsf{b}\cdot\mathsf{c}\,. \tag{20}$$

Notice that here two different gradients are implied: the one in the initial coordinates $\partial_{\mathbf{q}|\mathbf{p}}$ and the one in the final coordinates ∇, which is roughly $\partial_{\mathbf{q}|p,\varphi,\mathsf{c}}$, but which takes into account the necessary connection for c and the possible connection for φ. When acting on functions of $\mathbf{q}$ only, e.g. in the right-hand side of Eq. (20), they are equal, but in general they are not. What we call the physical connection is the one that makes them equal. For instance for the coordinate φ, it is defined by

$$\nabla\varphi = \partial_{\mathbf{q}|\mathbf{p}}\varphi\,.$$

The physical relevance of this connexion can be viewed in the components V_i of the velocity vector field, which are defined by the relation $\dot{f} = \mathbf{V}_i\cdot\partial_i f$ for any function f of the phase-space. Because of the non-trivial connexion, the components of the velocity vector

field $\mathbf{V}_i$ are different from the components of the velocity $\dot{\mathbf{z}}_i$, where $\mathbf{z} = (\mathbf{q}, p, \varphi, \mathsf{c})$ is a vector grouping all the coordinates. They are related by

$$\dot{\mathbf{z}}_i = \tfrac{d}{dt}\mathbf{z}_i = \sum_j \mathbf{V}_j \cdot \partial_j \mathbf{z}_i = \mathbf{V}_i + \mathbf{K}_i \,, \tag{21}$$

where $\mathbf{K}_i := \sum_j (1 - \delta_{ij}) \mathbf{V}_j \cdot \partial_j \mathbf{z}_i$ is a connexion term (be careful, the index i is not summed, although repeated), since it does not contribute when $\partial_j \mathbf{z}_i = 0$ for $i \neq j$. Here, the connection is involved only when a gradient acts on the coordinates c and possibly φ, and Eq. (21) is simply a way to write

$$\tfrac{d}{dt}\mathsf{c} = (\partial_t + \dot{\mathbf{q}} \cdot \nabla)\mathsf{c} \,,$$

and the same formula with c replaced by φ.

With the physical connexions (20) and (5) for the pitch angle and the gyro-angle, the components of the velocity vector field are given by

$$\mathbf{V}_i := \begin{pmatrix} \mathbf{p} \\ \frac{e\mathbf{E} \cdot \mathbf{p}}{p} \\ +\frac{e\mathbf{E}}{p \sin\varphi} \cdot \left(\cos\varphi \ \frac{\mathbf{p}}{p} - \mathsf{b} \right) \\ -\frac{eB}{m}\mathsf{a} + \frac{e\mathbf{E} \cdot \mathsf{a}}{p \ \sin\varphi}\mathsf{a} \end{pmatrix} .$$

They perfectly agree with the physical force, which is just the Lorentz force. Especially, the limit where there is no electric field $\mathbf{E} = 0$ is expressive: there remain only the velocity $\mathbf{V}_{\mathbf{q}} = \mathbf{p}/m$ and the Larmor gyration $\mathbf{V}_{\mathsf{c}} = -eB\mathsf{a}/m$. All the additional terms in the components φ and c of equations (2), which do not come from the physical dynamics but from the magnetic geometry, are absorbed in the connection. This is satisfactory since the role of the connexion is precisely to encode the change of the momentum coordinates through spatial displacement, as a result of the magnetic geometry.

When using the coordinate θ, the geometric contributions in Eq. (4) can not be absorbed in a connection contribution, since the scalar coordinate θ is introduced to make the coordinate system trivial, and hence to have flat connection. Providing θ with the corresponding connection would amount to using the intrinsic approach, with an additional detour by the gauge e_1.

The dynamics of the gyro-angle can be reinterpreted in this light, in relation with[19,21]. In the dynamics (4) of the gyro-angle θ, only the first and last terms are contributions due to the physical dynamics. The second term corresponds to the so-called "adiabatic phase" in the case considered by[19]; it comes from the magnetic term in the physical connection

(5), related to the definition for c to be physically the unit vector of the perpendicular momentum; it is induced by the change of the projection as a result of the change of the magnetic field (through spatial displacement); thus it concerns also the intrinsic gyro-angle c, e.g. in Eq. (2) or Eq. (5), and it is expected to be adiabatic only for the specific case considered by[19], but not for a general (inhomogeneous) strong magnetic field, which is confirmed by[21]. As for the "geometric phase", i.e. the third term in Eq. (4), it is actually a gauge phase, since it is purely related to the choice of gauge, and hence is absent from the intrinsic dynamics (2) or connection (5).

With the full physical connection (5) and (20), the coordinates c and φ are not independent of the variable $\mathbf{q}$, but they behave exactly as the components of a vector $\mathsf{v} := \mathsf{b}\cos\varphi + \mathsf{c}\sin\varphi$ that is independent of $\mathbf{q}$, i.e. that has flat connection:

$$\begin{aligned}\nabla\mathsf{v} &= \nabla(\cos\varphi\mathsf{b} + \sin\varphi\mathsf{c})\\ &= \cos\varphi\nabla\mathsf{b} + \sin\varphi\ \nabla\mathsf{c} + \nabla\varphi\ (-\mathsf{b}\sin\varphi + \mathsf{c}\cos\varphi) = 0\,.\end{aligned}$$

Actually, this computation shows that the flat connection $\nabla\mathsf{v} = 0$ is obtained if and only if the connection is the full physical one. The vector v stands for the unit vector of the momentum

$$\mathsf{v} := \tfrac{\mathbf{p}}{p}\,. \tag{22}$$

As a consequence, and as expected from the physical intuition, the commutator of gradients with this connexion is zero:

$$[\nabla_i, \nabla_j] = 0\,,$$

which is easily verified by direct computation and traduces that after one loop in configuration space, both the momentum and the magnetic field come back to their initial value.

All the same, even with this connection, the non-triviality of the fibre bundle for a general magnetic geometry should imply non-zero commutators. This is true, but in the gauge-independent approach, the space defined by the bundle is not the space of all $(\mathbf{q}, \mathsf{c})$, but the whole phase-space $(\mathbf{q}, p, \varphi, \mathsf{c})$. All the conclusions above apply, but replacing the phrase "gradients operators" by the phrase "basic derivative operators $\partial_{\mathbf{z}} = (\nabla, \partial_p, \partial_\varphi, \partial_\theta)$". Non-zero commutators are indeed present between these operators, as is shown in Eq. (23),

where non-trivial commutators of elementary derivatives are given for a general connection both for the gyro-angle $\mathbf{R}_g := \nabla \mathsf{c}\cdot\mathsf{a}$ and for the pitch-angle $\mathbf{R}_\varphi := \nabla\varphi$. This includes as special cases each of the four choices of connexion previously mentioned: the connexion (6) for the gauge-dependent case with coordinate θ; the physical connexion (5) for c; the general connexion (7) for c; and the full physical connection (5) and (20) for c and φ.

$$\begin{aligned}
[\nabla_i, \nabla_j] = {} & \nabla_i \mathsf{b}\cdot\mathsf{b}\times\nabla_j\mathsf{b}\ \partial_\theta \\
& -\Big(\nabla_i(\mathbf{R}_g)_j - \nabla_j(\mathbf{R}_g)_i\Big)\ \partial_\theta \\
& +\Big(\nabla_i(\mathbf{R}_\varphi)_j - \nabla_j(\mathbf{R}_\varphi)_i\Big)\ \partial_\varphi\,, \\
[\partial_p, \nabla] = {} & -\partial_p\mathbf{R}_g\ \partial_\theta + \partial_p\mathbf{R}_\varphi\ \partial_\varphi\,, \\
[\partial_\varphi, \nabla] = {} & -\partial_\varphi\mathbf{R}_g\ \partial_\theta + \partial_\varphi\mathbf{R}_\varphi\ \partial_\varphi\,, \\
[\partial_\theta, \nabla] = {} & -\partial_\theta\mathbf{R}_g\ \partial_\theta + \partial_\theta\mathbf{R}_\varphi\ \partial_\varphi\,.
\end{aligned} \tag{23}$$

In practical case, $\mathbf{R}_g$ does not depend on p, because the gyro-angle comes from the splitting of the coordinate v into the pitch-angle and the gyro-angle *via* the magnetic geometry $\mathbf{B}(\mathbf{q})$, in which the coordinate p plays no role.

Eq. (23) clearly emphasises the crucial role of the anholonomic magnetic term $\nabla\mathsf{b}\cdot\mathsf{b}\times\mathsf{b}'$, which is the only affine term in the connexion. For the minimal connexion, it is the only non-zero term, which indeed simplifies computations. As for the full physical connexion, it cancels this term, and also the whole commutator of gradients, but the two commutators $[\partial_\varphi, \nabla]$ and $[\partial_\theta, \nabla]$ become non-zero, since for instance $\partial_\varphi\mathbf{R}_g \neq 0$, and $\partial_\theta\mathbf{R}_\varphi \neq 0$.

With the general setting considered in Eq. (23), one can look for a connection that would make all commutators zero. This would provide a splitting of the vector v into proper scalar coordinates both for the pitch-angle and the gyro-angle, i.e. coordinates which fit with the action of commuting derivative operators. These quantities would be defined from the value of the quantity φ and c at one point in phase space through parallel transportation by the commuting derivative operators.

A solution is expected not to be generally possible since a scalar coordinate for the gyro-angle means that the circle bundle is trivial. The goal is to identify existence condition for the desired coordinate system. Indeed, the free 4-dimensional connexion function of phase-space $\big(\mathbf{R}_g(\mathbf{z}), \mathbf{R}_\varphi(\mathbf{z})\big)$ opens new possibilities. One can consider using this larger and

intrinsic freedom to obtain more complete results than with the restricted gauge-dependent framework, whose freedom corresponds only to the 1-dimensional gauge function of position space $\psi(\mathbf{q})$ in Eq. (8).

The last three rows in Eq. (23) imply that a solution $(\mathbf{R}_g, \mathbf{R}_\varphi)$ must not depend on φ, nor p, nor c, hence it must be purely position-dependent. In addition, the third row of Eq. (23) implies that $\mathbf{R}_\varphi$ must be curl-free, and as it is useless, it can be set to zero. As for the first two rows, they imply that $\mathbf{R}_g$ must cancel the anholonomy term, which is purely position-dependent. Thus, the equation for the desired connexion is

$$0 = \nabla_i \mathsf{b} \cdot \mathsf{b} \times \nabla_j \mathsf{b} - \nabla_i (\mathbf{R}_g)_j + \nabla_j (\mathbf{R}_g)_i \,,$$

which can be rewritten

$$\nabla \times \mathbf{R}_g = \mathbf{N} \,. \tag{24}$$

This equation is reminiscent of the usual relation (9) in the gauge-dependent approach, but they are different both in their origin and in their meaning. On the one hand, Eq. (24) is obtained without appealing to the idea of a gyro-gauge nor its associated gyro-angle θ. Starting from the structure of the manifold defined by the physical coordinates $(\mathbf{q}, p, \varphi, \mathsf{c})$, we are looking for an arbitrary scalar gyro-angle coordinate, which might not be related to a choice of gauge; more precisely we are looking for a connexion corresponding to this coordinate. On the other hand, relation (24) is a necessary and sufficient condition for the existence of a scalar gyro-angle, whereas in previous works, the analogous relation (9) only appeared as a consequence of the existence of a gauge.

When a solution $\mathbf{R}_s$ of Eq. (24) exists, the scalar gyro-angle coordinate is defined from parallel transportation with the derivative operators defined by the associated connexion $\mathbf{R}_s$. This parallel transportation results in a (trivializing) global section of the circle bundle, which in turn provides a zero for measuring the gyro-angle. Although analogous to a traditional gyro-gauge-fixing, this is more general, because it implies a section of the whole bundle $(\mathbf{q}, p, \varphi, \mathsf{c})$, not a section of the restricted bundle $(\mathbf{q}, \mathsf{c})$ over the position space. Now, the important point in the analysis above is that $\mathbf{R}_s$ depends only on the position; so, the parallel transportation actually results in a section of the restricted circle bundle over the position space. This means that a scalar coordinate always defines a gyro-gauge, which is the reciprocal of the well-known property, saying that a gyro-gauge provides a scalar gyro-angle, which was considered in previous works and in the previous section.

As a consequence, condition (24) is also a necessary and sufficient condition for a global gauge to exist. Here, it is obtained in a direct argument on commuting derivatives. This is very different from the work[15], where the existence of a global gauge was studied, but the proof for the necessary condition used an auxiliary property and the proof for sufficiency required "a lengthy digression into the theory of principal bundles and characteristics". Finally, their condition for the existence of a gauge is slightly different from ours, but they are equivalent. Their condition states that through the boundary S of any hole inside the spatial domain, the vector field $\mathbf{N}$ has no net flux:

$$\oint_S \mathbf{N}\cdot d\mathbf{S} = 0\,,$$

and this is the boundary condition for the solvability of Eq. (24), since $\nabla\cdot\mathbf{N} = 0$.

VI. A GAUGE-INDEPENDENT AND UNCONSTRAINED COORDINATE SYSTEM

The previous sections were concerned with difficulties of the traditional coordinate θ. They appeared as related to properties of the basic derivative operators involved in guiding-center theory, which do not fit with the intrinsic coordinate system. In this sense, the philosophy of the gauge-independent approach is to consider separately the coordinate system and the elementary derivative operators, and to avoid putting in the coordinate system some properties which actually concern the derivative operators.

The resulting formalism may seem more complicated than expected, because of the constrained coordinate system, with the associated subtleties about covariant derivatives, non-zero commutators, non-closed basis of 1-forms, etc. One can consider going one step further, and removing these subtleties from the coordinate system. In one way or another, they are unavoidable in the theory, since they result from properties of the non-trivial circle bundle implied by the fast guiding-center coordinate. But they concern the derivative operators, not the coordinate system, so that this last can be made both gauge-independent and unconstrained.

Since the constrained coordinate system comes from the gyro-angle coordinate, with its $\mathbb{S}^1$ fibre-bundle, in order to get rid of the associated formalism, the principle is again to come

back to more primitive coordinates, namely by avoiding the splitting of the momentum into the pitch-angle and the gyro-angle. Then, the unit vector of the momentum $\mathsf{v} := \frac{\mathbf{p}}{p}$ is kept as a single two-dimensional coordinate, as was approached by the results of the previous section.

Segregating the two coordinates is needed at the end of the guiding-center reduction, when the gyro-angle is removed from the dynamics, to obtain the slow reduced dynamics. But at that point, the fibre-bundle and the constrained coordinate c are also removed. In the course of the reduction process, the splitting is not needed. What is needed is a basis of 1-forms and derivative operators that fits with the separation of scales, for instance to decompose the transformation of the vector v between the fast gyro-angle and the slow pitch-angle, or to decompose the change of spatial coordinate into its components transverse and parallel to the magnetic field.

So, the method of using intrinsic coordinates and defining derivative operators adapted to the purpose can be applied. The essential novelty is that the definition space for the coordinate v is the sphere $\mathbb{S}^2$, whose immersion in $\mathbb{R}^3$ is independent of the spatial position. So, a trivial connection is available, and it corresponds both to the minimal and to the physical connection, since the definition (22) of v does not depend on the position.

Notice that since this coordinate is a two-dimensional vector immersed in $\mathbb{R}^3$, its variations are constrained and the operator ∂_{v} as well as the 1-form $d\mathsf{v}$ are purely transverse: $\mathsf{v}\cdot\partial_{\mathsf{v}} = \mathsf{v}\cdot d\mathsf{v} = 0$. But the coordinate system is not constrained any more: the coordinates are independent of each other and the basic differential operators and 1-forms $\partial_{\mathbf{q}}$, ∂_p, ∂_{v}, $d\mathbf{q}$, dp, $d\mathsf{v}$, behave trivially, which makes the practical treatment similar to standard coordinate systems.

For the purpose of the guiding-center reduction, the splitting between the pitch-angle and the gyro-angle is implemented in the basis of vector fields and 1-forms: ∂_{v} and $d\mathsf{v}$ are decomposed to distinguish their contributions in the azimuthal direction a (corresponding to the variable θ), and in the elevation direction $\mathsf{a} \times \mathsf{v}$ (corresponding to the variable φ). For instance, the operator ∂_{v} can be decomposed as $\left(\mathsf{v} \times \mathsf{b}\cdot\partial_{\mathsf{v}}, \frac{(\mathsf{b}\times\mathsf{v})\times\mathsf{v}}{\sin\varphi}\cdot\partial_{\mathsf{v}}\right)$, in order to agree with the traditional operators $(\partial_\theta, \partial_\varphi)$. Alternatively, the second operator can be chosen $-\sin\varphi\ (\mathsf{b}\times\mathsf{v})\times\mathsf{v}\cdot\partial_{\mathsf{v}} = \partial_\phi$, in order to fit with the variable $\phi := \cot\varphi$ which made formulae polynomials[16–18]. In a simpler way, it can be chosen just $(\mathsf{b}\times\mathsf{v})\times\mathsf{v}\cdot\partial_{\mathsf{v}}$. This arbitrariness

in the choice of a basis for vector fields is similar to the connection freedom in previous sections, but it is not the same, since it concerns the splitting of the operator ∂_{v} rather than the definition of the gradient operator.

This splitting procedure is only a generalization of what was already done for the position space in previous works, where the three-dimensional quantity $\mathbf{q}$ was kept as a coordinate but the gradient ∇ and the differential form $d\mathbf{q}$ were split into scalar components suited to the derivation, namely their components parallel to a, b, and c.

It is straightforward to verify that guiding-center reductions work as usual with this formalism. In a similar way as what was observed when going from the coordinate θ to the coordinate c, the introduction of the coordinate v removes all the intricacies from the coordinate system and confines them to the basis of vector fields of the theory. The only difference is that here, the basis is not induced by the coordinate system, whose associated basic derivative operators are now trivial, but it is induced by the purposes of the guiding-center reduction, such as the separation of scales. All the subtleties mentioned in the previous sections remain present, but they are encoded in the properties of the chosen basis. For instance, when the basis is chosen $\left(\mathsf{v} \times \mathsf{b}\cdot\partial_{\mathsf{v}}, \frac{(\mathsf{b}\times\mathsf{v})\times\mathsf{v}}{\sin\varphi}\cdot\partial_{\mathsf{v}}\right)$, the elementary differential operators are equal to their counterparts $(\partial_\theta, \partial_\varphi)$ in the formalism of the previous section, with the "full physical connexion" (5) and (20).

One could consider going one step further and keeping all the momentum coordinates (p, v) as a single coordinate $\mathbf{p}$. This is unsure to be relevant, since the coordinate p actually plays no role in the introduction of the gyro-angle, as we mentioned previously. In addition, keeping the coordinate $\mathbf{p}$ does not seem convenient for practical computations. When the norm of the momentum is kept in the reduced coordinates, as in[17], then p is unchanged by the guiding-center transformation and it is useless to group it with v into one single coordinate $\mathbf{p}$. Most often, the coordinate p is replaced by the constant of motion conjugated to the gyro-angle, the magnetic moment $\bar{\mu}$, then the coordinate p is usually changed to μ in a preliminary step, for the remaining transformation to be near-identity. In this case, the splitting of the coordinate $\mathbf{p}$ into p and v is essential. Last, separating the variables p and v is interesting for dimensional reasons, because v is dimensionless and then only one of the momentum coordinates, p, is dimensional (see[18] for an analysis of the practical consequences of such a dimensional argument).

VII. CONCLUSION

The gauge-independent approach of guiding-center theory actually resolves the difficulties associated with the usual gyro-angle coordinate. The use of the physical gyro-angle as a coordinate removes the non-global existence, the gauge dependence and the anholonomy from the coordinate system, which then agrees both with the physical state and with the mathematical structure of the system, a non-trivial circle bundle.

This physical gyro-angle is constrained and position-dependent, which implies the presence of a covariant gradient, encoding the geometry of the bundle. The corresponding connexion involves a freedom, which is the intrinsic counterpart of the gauge arbitrariness but is much larger than it and very different; especially, it does not affect the coordinate system and is just a choice in the basis of vector fields of the theory.

Because of the larger freedom, relevant choices become available. For instance, the connexion can be chosen so as to fit with the physical definition of the gyro-angle. Alternatively, it can be set to zero; this minimal connexion simplifies computations and removes the arbitrary terms from the theory; it was found to be underlying in previous results and can be considered as the natural geometrical connexion. Both choices do not correspond to a gauge fixing, but to a connexion fixing, which shows that the intrinsic formulation is needed to capture the physics and the mathematics of the guiding-center system. This is also emphasized by the fact that the issue about non-global existence not only disappears, but has no counterparts in the intrinsic formulation.

Both the physical and the minimal covariant gradients have non-zero commutators, which are the counterparts of the anholonomy of the gauge-dependent approach. Again, they do not concern the coordinate system but the basic derivative operators. A third choice of connexion was identified, by giving to the pitch-angle the connection naturally induced by its definition from the physical momentum. Then, covariant gradients do commute, but other non-zero commutators appear in phase-space, which traduce the non-triviality of the circle bundle defined by the gyro-angle.

In this framework, existence conditions for a splitting of the momentum into scalar coordinates for the pitch-angle and for the gyro-angle can be studied. The point of view is simplified and more general compared to the study of the existence of a gyro-gauge. The resulting condition is just the invertibility of a curl on a divergenceless vector field, which

corresponds to boundary conditions on this vector field, in agreement with previous results.

As a result, the gauge-independent formulation emphasizes the intrinsic properties underlying the initial difficulties of the traditional gyro-angle. It mainly replaced the issues in the coordinate system by regular, although non-trivial, properties of the basic derivative operators.

The issues were related to the requirement for the coordinates and the basic derivative operators to behave trivially. This is possible only when the circle bundle is trivial, which can be obtained only locally for a general magnetic geometry and keeps the anholonomy question. To agree with the physical system, the basic derivative operators must have non commuting properties, which in turn means that there do not exist trivializing coordinates and that a constrained coordinate has to be used for the gyro-angle.

An idea underlying the intrinsic approach is that perturbation theory needs adapted derivative operators, but not necessarily adapted coordinates, so that the coordinate system can be chosen as it is intrinsically, i.e. as it comes primitively. This idea can be generalized to avoid introducing the gyro-angle coordinate, with the associated intricacies of a constrained coordinate system. Keeping the gyro-angle and the pitch-angle as the single initial coordinate, which is the unit vector of the momentum, the $\mathbb{S}^1$-bundle is replaced by an $\mathbb{S}^2$-bundle, which can be made (and is naturally) trivial, i.e. with no covariant derivatives (trivial connexion) nor any non-zero commutator. The resulting coordinate system is both intrinsic and unconstrained. For guiding-center perturbation theory, an adapted basis of derivative operators is defined, which encodes all the intrinsic properties of the circle bundle associated to the gyro-angle.

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